MATH 245 S21, Exam 1 Solutions

2. Prove or disprove: If a is an even integer, then $\frac{a^3}{4}$ must be even.

The statement is true, and here is a direct proof. Suppose that a is an even integer. Now, there must be some integer b with $a = 2b$. Now, $\frac{a^3}{4} = \frac{(2b)^3}{4} = \frac{8b^3}{4} = 2b^3$. Note that b^3 must be a integer, being the product of integers b, b, b. Hence, $\frac{a^3}{4}$ is twice an integer, so it must be even.

3. Let p, q be propositions. Simplify the following expression as much as possible (where only basic propositions are negated): $\neg(p \leftrightarrow q)$.

Applying Theorem 2.17 to $p \leftrightarrow q$, we find that our proposition is equivalent to $\neg((p \to q) \wedge (q \to p))$. We apply De Morgan's Law to get $(\neg(p \to q)) \vee (\neg(q \to p))$. We can now finish in two different ways.

METHOD 1: Apply Thm 2.16 twice, to get $(p \wedge \neg q) \vee (q \wedge \neg p)$.
METHOD 2: Apply Conditional Interpretation twice, to get $(\neg(q \vee \neg p)) \vee (\neg(p \vee \neg q))$. Apply De Morgan's Law twice to get $((\neg q) \wedge \neg\neg p) \vee ((\neg p) \wedge \neg\neg q)$. Lastly, apply Double Negation twice to get $((\neg q) \wedge p) \vee ((\neg p) \wedge q)$.

4. Let p, q, r, s be propositions. Prove $p \to (q \vee r), q \to s, r \to s \vdash p \to s$.

We begin by assuming that$p \to (q \vee r), q \to s$, and $r \to s$ are all true. Most proofs will have cases.
SOLUTION 1: We break into cases depending on if s is true or false. If s is true, then $p \to s$ is true trivially. If s is false, then $\neg r$ by modus tollens with $r \to s$. Also, if s is false, then $\neg q$ by modus tollens with $q \to s$. By conjunction, $(\neg r) \wedge (\neg q)$. By De Morgan's Law, $\neg(r \vee q)$. Then $\neg p$ by modus tollens with $p \to (q \vee r)$. Finally, $p \to s$ is true vacuously.

SOLUTION 2: Applying conditional interpretation to $p \to (q \vee r)$, we get $q \vee r \vee \neg p$. This gives three cases. Case q: We get s by modus ponens with $q \to s$, so $p \to s$ is true trivially. Cased r: we get s by modus ponens with $r \to s$, so $p \to s$ is again true trivially. Case $\neg p$: Now $p \to s$ is true vacuously.

SOLUTION 3: It is also possible to do this with a huge truth table (9 columns, 16 rows!). Unless you don't mind spending half the exam time on one problem, this is not recommended.

5. Prove or disprove: For all $p \in \mathbb{N}$, if p^2 is prime then p is prime.

The statement is true. All correct solutions must consider $p = 1$ separately from $p > 1$, and prove that p^2 is not prime using Definition 1.16.

SOLUTION 1: vacuous proof. We will prove that p^2 is never prime. There are two cases, either $p = 1$ or $p > 1$. If $p = 1$ then $p^2 = 1$, and so p^2 is not prime (primes must be integers at least 2). If instead $p > 1$, then there exists an integer p with $1 < p < p^2$ and $p|p^2$, so p^2 is composite and hence not prime.

SOLUTION 2: contrapositive proof. Suppose that p is not prime. Hence either $p = 1$, or there is some $a \in \mathbb{N}$ with $1 < a < p$ and $a|p$. In the case $p = 1$, then $p^2 = 1$, so p^2 is not prime. In the other case, there must be some $b \in \mathbb{N}$ with $p = ab$. Then, we have $p^2 = a(bp)$. Hence $a|p^2$, and also $1 < a < p^2$, so p^2 is composite and hence not prime. In both cases, p^2 is not prime.

6. Prove or disprove: $\forall x \in \mathbb{N},\ \exists y \in \mathbb{Q},\ |x - y| = |y|$.

The statement is true. Let $x \in \mathbb{N}$ be arbitrary. Choose $y = \frac{x}{2}$, which must be in $\mathbb{Q}$. Now $|x - y| = |x - \frac{x}{2}| = |\frac{x}{2}| = |y|$. Hence $|x - y| = |y|$.

7. Prove or disprove: $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ |x - y| = |y|$.

The statement is false. Choose $x = 1$, and let $y \in \mathbb{N}$ be arbitrary. Because $y \geq 1$, $1 - y \leq 0$, so we have $|x - y| = |1 - y| = -(1 - y) = y - 1$. But also $|y| = y$, since $y \geq 0$. Now $y \neq y - 1$, hence $|x - y| \neq |y|$.

Comparing the last two questions, we see that the difference is whether y is allowed to be rational or not. If $x = 1$, we need to make $y = \frac{1}{2}$ to make the two absolute values equal.